

On the Generators in the Category of Actions of Pomonoids on Posets and its Slices

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Abstract

Let S be a pomonoid, in this paper, $\mathbf{Pos}\text{-}S$, the category of S -posets and S -poset maps, is considered. First, we characterize some pomonoids on which all projectives in this category are generator or free. Then, we study regular injectivity and weakly regularly d -injectivity which lead to some homological classification results for pomonoids. Among other things, we get some relationships between regular injectivity in the slice category $\mathbf{Pos}\text{-}S/B_S$ and generators or cyclic projectives in $\mathbf{Pos}\text{-}S$.

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1 Introduction and Preliminaries

Although there exist many papers which investigate various properties of generator S -acts, among them, there seems to be known very little on generators S -posets. V. Laan investigated some properties of generator S -posets in [11]. Continuing this study, in this paper, after some introductory results in section 1, we attempt in section 2 to collect new results on generators in $\mathbf{Pos}\text{-}S$ to reply on the questions of homological classification of pomonoids.

\mathcal{M} -injective objects in the slice category \mathcal{C}/B , for any B in \mathcal{C} , form the right part of a weak factorization system that has morphisms of \mathcal{M} as the left part (see [2]). Here, we consider the same case in the slice category $\mathbf{Pos}\text{-}S/B_S$ of right S -poset maps over B_S , where B_S is an arbitrary S -poset. In section 3, we first, find some conditions for when all generators are regular d -injective or weakly regularly injective. Finally, we prove that every Emb-injective object in $\mathbf{Pos}\text{-}S/B_S$ is split epimorphism. By this fact, we received to some generators and cyclic projectives in $\mathbf{Pos}\text{-}S$ and in the category of actions of endomorphism pomonoids on posets.

For the rest of this section, we give some preliminaries about S -acts, S -posets and slice category which we will need in the sequel.

Let S be a monoid with identity 1. Recall that a (right) S -act is a set A equipped with a map $\mu : A \times S \rightarrow A$ called its action, such that, denoting $\mu(a, s)$ by as , we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. The category of all S -acts, with action-preserving (S -act) maps ($f : A \rightarrow B$ with $f(as) = f(a)s$, for $s \in S, a \in A$), is denoted by $\mathbf{Act}\text{-}S$. For instance, take any monoid S and a non-empty set A . Then A becomes a right S -act by defining $as = a$ for all $a \in A, s \in S$, we call that A an S -act with trivial action. Clearly S itself is an S -act with its operation as the action. For more information about S -acts see [10].

A monoid S is said to be a *pomonoid* if it is also a poset whose partial order \leq is compatible with the binary operation, i.e., $s \leq t, s' \leq t'$ imply $ss' \leq tt'$ (see [3]). In this paper S denotes a pomonoid with an arbitrary order, unless otherwise stated.

On a monoid S we define the following relations: for every $s, t \in S$

1. $s\mathcal{R}t$ if $sS = tS$.
2. $s\mathcal{J}t$ if $SsS = StS$.
3. $s\mathcal{D}t$ if there exists $u \in S$ with $sS = uS$ and $St = Su$.

These relations are called Green's relations on S (see [10]). Here, we consider these notions for pomonoid S and supply some suitable results.

Let S be a pomonoid. A (*right*) S -poset is a poset A which is also an S -act whose action $\mu : A \times S \rightarrow A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S -posets with action preserving monotone maps is denoted by $\mathbf{Pos}\text{-}S$. Clearly S itself is an S -poset with its operation as the action. Left S -poset can be defined analogously (see [4]). A left T -poset (${}_TA$) and right S -poset (A_S) is called T - S -biposet (and denoted by ${}_TA_S$) when $(ta)s = t(as)$ for every $s \in S, t \in T$ and $a \in A$. We remind the following results from [11]:

- (i) For every A_S in $\mathbf{Pos}\text{-}S$, consider the set $\text{End}(A_S) = \mathbf{Pos}_S(A, A)$ as a pomonoid with respect to composition and pointwise order. Also, we define the left $\text{End}(A_S)$ -action on A by $f \cdot a = f(a)$, for every $a \in A, f \in \text{End}(A_S)$, so that one has ${}_{\text{End}(A_S)}A_S$.
- (ii) The following mappings are pomonoid homomorphisms,

$$\begin{aligned} \rho : S &\rightarrow \text{End}({}_TA); & s &\mapsto \rho_s \\ \lambda : T &\rightarrow \text{End}A_S); & t &\mapsto \lambda_t, \end{aligned}$$

where $\rho_s : {}_TA \rightarrow {}_TA, a \mapsto as$ and $\lambda_t : A_S \rightarrow A_S, a \mapsto ta$ are morphisms in $T\text{-}\mathbf{Pos}$ and $\mathbf{Pos}\text{-}S$, respectively.

- (iii) For every T - S -biposet ${}_TA_S$ recall that if $B \in \mathbf{Pos}\text{-}S$ then the set $\mathbf{Pos}_S(B, A)$ of all S -poset maps from B_S to A_S is an object in $T\text{-}\mathbf{Pos}$ with the action defined by

$$t \cdot f = \lambda_t f$$

for every $f \in \mathbf{Pos}_S(B, A), t \in T$. Consequently, we have a functor

$$\mathbf{Pos}_S(-, A) : \mathbf{Pos}\text{-}S \rightarrow T\text{-}\mathbf{Pos}$$

by taking

$$\mathbf{Pos}_S(-, A)(P) = \mathbf{Pos}_S(P, A)$$

for every $P \in \mathbf{Pos}\text{-}S$.

An S -poset G_S is a generator in the category $\mathbf{Pos}\text{-}S$ if for any distinct S -poset maps $\alpha, \beta : X_S \rightarrow Y_S$ there exists an S -poset map $f : G_S \rightarrow X_S$ such that $\alpha f \neq \beta f$.

Given a category \mathcal{C} and an object B of \mathcal{C} , one can construct the *slice category* \mathcal{C}/B (read: \mathcal{C} over B): objects of \mathcal{C}/B are morphisms of \mathcal{C} with codomain B , and morphisms in \mathcal{C}/B from one such object $f : D \rightarrow B$ to another $g : E \rightarrow B$ are commutative triangles in \mathcal{C}

$$\begin{array}{ccc} D & \xrightarrow{h} & E \\ & \searrow f & \swarrow g \\ & B & \end{array}$$

i.e, $gh = f$. The composition in \mathcal{C}/B is defined from the composition in \mathcal{C} , in the obvious way. It means paste triangles side by side.

A poset is said to be *complete* if each of its subsets has an infimum and a supremum, in particular, a complete poset is bounded, that is, it has the least (bottom) element \perp and the greatest (top) element \top .

2 Some Homological Classifications for Pomonoids by Generators in $\mathbf{Pos}\text{-}S$

In this section, we discuss about the generators and projective generators in $\mathbf{Pos}\text{-}S$ and give some new properties of them.

As we mentioned in the introduction, generators have already been characterized in [11] for the category $\mathbf{Pos}\text{-}S$, such as the following two first propositions.

Proposition 2.1. *Cyclic projectives in $\mathbf{Pos}\text{-}S$ are precisely retracts of S_S .*

Proposition 2.2. *An S -poset A_S is a cyclic projective generator in $\mathbf{Pos}\text{-}S$ if and only if $A_S \cong eS_S$ for an idempotent $e \in S$ with $e\mathcal{I}1$.*

It is clear that the previous proposition lead to the following result:

Proposition 2.3. *Let S be a commutative pomonoid. Then all cyclic projective generators in $\mathbf{Pos}\text{-}S$ are isomorphic to S_S .*

We need the following proposition from [14], as a characterization of cyclic projective S -posets.

Proposition 2.4. *Let A_S be an S -poset and $a \in A$. Then the following statement are equivalent:*

- (i) aS_S is projective.
- (ii) $aS_S \cong eS_S$ for some idempotent $e \in S$.

In [14] the authors found a decomposition theorem for projective S -posets. In the following, by this fact and Proposition 2.2, we generalize a description of projective generator S -acts to S -posets. The proof is almost identical to that for Proposition 3.18.5 from [10].

Theorem 2.5. *Every S -poset P_S is projective generator if and only if $P_S = \coprod_{i \in I} P_i$ where $P_i \cong e_i S$ for every $i \in I$, and at least one P_j , $j \in I$ is a generator with $e_j \mathcal{J}1$.*

Notice that for every pomonoid S and an idempotent $e \in S$, the sub S -poset eS_S of S_S is projective according to Proposition 2.4, but it is not a generator because $e\mathcal{J}1$ is not necessary true.

Next, by Theorem 2.5 we get the following result.

Theorem 2.6. *For any pomonoid S the following statements are equivalent:*

- (i) *All projective right S -posets are generators in $\mathbf{Pos}\text{-}S$.*
- (ii) *All cyclic projective right S -posets are generators in $\mathbf{Pos}\text{-}S$.*
- (iii) *$e\mathcal{J}1$ for every idempotent $e \in S$.*

Recall [5] that a left *poideal* of a pomonoid S is a (possibly empty) subset I of S if it is both a monoid left ideal ($SI \subseteq I$) and a down set ($a \leq b, b \in I$ imply $a \in I$). For example

$$\downarrow eS = \{t \in S : \exists s \in S, t \leq es\}$$

is a cyclic poideal of S , for every idempotent $e \in S$.

In the following we shall characterize pomonoids that all their principal right poideals are generators.

Proposition 2.7. *Let S be a pomonoid and $e \in S$ with $e^2 = e$. If the cyclic projective sub S -poset eS_S of S_S is a generator in $\mathbf{Pos}\text{-}S$, then $\downarrow eS$ is also a generator.*

Proof. By assumption there exists an S -poset epimorphism (is exactly onto in $\mathbf{Pos}\text{-}S$) $f : eS_S \rightarrow S_S$. Define mapping $g : \downarrow eS \rightarrow S_S$ by $g(x) := f(ex)$ for every $x \in \downarrow eS$. It is easy to see that g is an S -poset map. Also, for every $s \in S$ there exists $u \in S$ such that $f(eu) = s$. Then we have

$$g(eu) = f(eeu) = f(eu) = s$$

This means that g is an epimorphism. By Theorem 2.1 in [11], we conclude $\downarrow eS$ is a generator, as we required. \square

Lemma 2.8. *Let S be a pomonoid and $z \in S$. If the principal right poideal $\downarrow zS$ is a generator in $\mathbf{Pos}\text{-}S$, then there exist $x, y \in S$ such that $1 \leq yx$, and $za \leq zb$, $a, b \in S$ implies $ya \leq yb$.*

Proof. Since $\downarrow zS$ is a generator in $\mathbf{Pos}\text{-}S$, by Theorem 2.1 of [11], there exists an epimorphism $g : \downarrow zS \rightarrow S_S$. Hence, there are $u \in \downarrow zS$ and $t \in S$ such that $u \leq zt$ and $g(u) = 1$. Let $y = g(z)$ and $x = t$. Then $yx = g(z)x = g(zx)$. Since $u \leq zx$, the monotone property of g implies $g(u) \leq g(zx)$. Consequently, $1 = g(u) \leq g(zx) = yx$. Now, suppose $za \leq zb$, $a, b \in S$. Then

$$ya = g(z)a = g(za) \leq g(zb) = g(z)b = yb.$$

\square

Proposition 2.9. *Let S be a pomonoid in which the identity element is the top element. If all poideals of S are generators then the sub S -poset eS_S of S_S is a generator in $\mathbf{Pos}\text{-}S$, for every idempotent $e \in S$.*

Proof. By hypothesis, for every idempotent $e \in S$, $\downarrow eS$ is a generator in $\mathbf{Pos}\text{-}S$. Then by Lemma 2.8, there exist $x, y \in S$ such that $1 \leq yx$ and $ea \leq eb$, $a, b \in S$, always implies $ya \leq yb$. In particular, since $e1 \leq ee$ then $y \leq ye$. Now, $1 \leq yx \leq yex$. Also, evidently, we have $yex \leq 1$. Consequently, $yex = 1$ this means that $e\mathcal{J}1$, equivalently eS_S is a projective generator by Proposition 2.2, as we needed. \square

Theorem 2.10. *Let S be a pomonoid in which the identity element is the top element. The following statements are equivalent.*

- (i) *All projective right S -posets are generators in $\mathbf{Pos}\text{-}S$.*
- (ii) *All cyclic projective right S -posets are generators in $\mathbf{Pos}\text{-}S$.*
- (iii) *$e\mathcal{J}1$ for every idempotent $e \in S$.*
- (iv) *All principal right poideals of S which are generated by an idempotent, are generators in $\mathbf{Pos}\text{-}S$.*

Proof. (i) \iff (ii) \iff (iii): It is Theorem 2.6.

(iii) \implies (iv). By Proposition 2.2 we get eS_S is a cyclic projective generator. Next, Proposition 2.7 shows that $\downarrow eS$ is a generator in $\mathbf{Pos}\text{-}S$.

(iv) \implies (iii). Consider the principal right poideal $\downarrow eS$ for every idempotent $e \in S$. By a similar proof of Proposition 2.9, the cyclic projective sub S -poset eS_S of S_S is a generator. Using Proposition 2.2, we conclude that $e\mathcal{J}1$. \square

By a *free S -poset on a poset P* [4] we mean an S -poset F together with a poset map $\tau : P \rightarrow F$ with the universal property that given any S -poset A and a poset map $f : P \rightarrow A$ there exists a unique S -poset map $\bar{f} : F \rightarrow A$ such that $\bar{f} \circ \tau = f$, i.e, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\tau} & F \\ & \searrow f & \downarrow \bar{f} \\ & & A \end{array}$$

commutes. Also S -poset F is given by $F = P \times S$ with componentwise order and action $(x, s)t = (x, st)$, for $x \in P$ and $s, t \in S$ (see [4]).

Example 2.11. Let S be a pomonoid generated by the elements e, k, k' and with discrete order such that $kk' = 1$ and $ek = k'$. Then eS_S is a cyclic projective generator in $\mathbf{Pos}\text{-}S$. But eS_S is not free (see Lemma 2.12 below).

In the following lemma we shall characterize idempotents of monoid S which generate free cyclic sub S -acts (see Proposition 3.17.17 of [10]) of S_S generalizes to the category of S -posets. Moreover, we conclude when projective (or cyclic projective) implies free in $\mathbf{Pos}\text{-}S$.

Lemma 2.12. *Let e be an idempotent of a pomonoid S . Then the sub S -poset eS_S of S_S is a free right S -poset if and only if $e\mathcal{D}1$.*

Theorem 2.13. *For any pomonoid S the following statements are equivalent:*

- (i) *All projective right S -posets are free.*
- (ii) *All projective generators in $\mathbf{Pos}\text{-}S$ are free.*
- (iii) *All cyclic projective right S -posets are free.*
- (iv) *$e\mathcal{D}1$ for every idempotent $e \in S$.*

Proof. (i) \implies (ii) is trivial.

(ii) \implies (iii). By Proposition 2.4, all cyclic projective S -posets are isomorphism to eS_S for some idempotent $e \in S$. Let $A = S_S \coprod eS_S$. By Proposition 2.5, A_S is a projective generator in $\mathbf{Pos}\text{-}S$. By hypothesis A_S is free which implies that eS_S is free.

(iii) \implies (i). By decomposition theorem in [14], every projective S -poset is isomorphic to a coproduct of cyclic projective S -posets which are free by assumption. Now we get the result.

(iii) \iff (iv). By characterization of cyclic projective S -posets in Proposition 2.4 and Lemma 2.12 we get the equivalence.

□

3 Regular Injectivity in $\mathbf{Pos}\text{-}S$ and $\mathbf{Pos}\text{-}S/B_S$ and Generators

Let \mathcal{C} be a category and \mathcal{M} a class of its morphisms. An object I of \mathcal{C} is called \mathcal{M} -injective if for each \mathcal{M} -morphism $h : U \rightarrow V$ and morphism $u : U \rightarrow I$ there exists a morphism $s : V \rightarrow I$ such that $sh = u$. That is, the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{u} & I \\ h \downarrow & \nearrow s & \\ V & & \end{array}$$

In particular, this means that, in the slice category \mathcal{C}/B , $f : X \rightarrow B$ is \mathcal{M} -injective if, for any commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ h \downarrow & & \downarrow f \\ V & \xrightarrow{v} & B \end{array}$$

with $h \in \mathcal{M}$, there exists an arrow $s : V \rightarrow X$ such that $sh = u$ and $fs = v$.

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ h \downarrow & \nearrow s & \downarrow f \\ V & \xrightarrow{v} & B \end{array}$$

Recall that regular monomorphisms (morphisms which are equalizers) in $\mathbf{Pos}\text{-}S$ (and also in $\mathbf{Pos}\text{-}S/B_S$) are exactly order-embeddings (see [4] and ([7])). In the following we consider Emb-injectivity in $\mathbf{Pos}\text{-}S$ and $\mathbf{Pos}\text{-}S/B_S$, where Emb is the class of all order-embeddings of S -posets.

Theorem 3.1. *All generators in $\mathbf{Pos}\text{-}S$ are Emb-injective if and only if all S -posets are Emb-injective.*

Proof. (\implies) Let A_S be an S -poset. Consider $A_S \times S_S$ which is a generator in $\mathbf{Pos}\text{-}S$ and so is Emb-injective. Hence A_S is Emb-injective.

(\impliedby) It is clear. \square

Note that the class of all embeddings of right poideals into S_S is a subclass of all down-closed embeddings, i.e. all embeddings $g : B \rightarrow C$ with the property that $g(B)$ is down-closed in C , and hence a subclass of all embeddings.

Definition 3.2. An S -poset of A_S is said (*principally*) *weakly regularly d -injective* if it is injective with respect to all embeddings of (principal) right poideals into S_S .

Proposition 3.3. *If all generators in $\mathbf{Pos}\text{-}S$ are weakly regularly d -injective then all S -posets are weakly regularly d -injective.*

Proof. Let A_S be an S -poset. Consider $A_S \times S_S$ which is a generator in $\mathbf{Pos}\text{-}S$ and so is weakly regularly d -injective. To show that A_S is

weakly regularly d -injective consider the following diagram

$$\begin{array}{ccc} I_S & \xrightarrow{u} & A_S \\ \downarrow i & & \\ S_S & & \end{array}$$

where I is a poideal of S . Define S -poset map $\bar{u} : I_S \rightarrow A_S \times S_S$ by $\bar{u}(s) = (u(s), s)$ for each $s \in S$. Hence, by assumption, there exists an S -poset map $v : S_S \rightarrow A_S \times S_S$ such that $vi = \bar{u}$. By composition with the projection $\pi_A : A_S \times S_S \rightarrow A_S$, we get that A_S is a weakly regularly d -injective. \square

Recall that for a pomonoid S an element $s \in S$ is called a *regular element* if there exists $t \in S$ such that $sts = s$. One calls S a *regular pomonoid* if all its elements are regular.

Theorem 3.4. *Let S be a pomonoid whose the identity element is the top element. Then the following statements are equivalent:*

- (i) *All S -posets are principally weakly regularly d -injective.*
- (ii) *All principal right poideals of S are principally weakly regularly d -injective.*
- (iii) *All generators in $\mathbf{Pos}\text{-}S$ are principally weakly regularly d -injective.*
- (iv) *S is a regular pomonoid.*

Proof. (i) \implies (ii) is trivial.

(i) \iff (iii) Proposition 3.1.

(ii) \implies (iv) For every $s \in S$, consider the down closed embedding $i : \downarrow sS \rightarrow S_S, x \mapsto x$. Then it has a left inverse f , since $\downarrow sS$ is principally weakly regularly d -injective. Now, taking $f(1) = z$, we have $z \leq st$ for some $t \in S$ and

$$s = f(s) = f(1)s = zs \leq sts.$$

Also, $sts \leq s$, since 1 is the top element of S . Therefore s is a regular element. This means that S is a regular pomonoid.

(iv) \implies (i) See Theorem 3.6 in [13]. \square

Recall from [5] that a pomonoid S which has no proper non-empty left (right) poideal is said to be left (right) *simple*.

Corollary 3.5. *If all generators in $\mathbf{Pos}\text{-}S$ are *Emb-injective* then S is left simple.*

Proof. From hypothesis and Proposition 3.1, we conclude that all complete S -posets are *Emb-injective*. Now Theorem 3.9 in [5] allow us to say that S is left simple. \square

Proposition 3.6. *For any pomonoid S the following statements are equivalent:*

- (i) *All generators in $\mathbf{Pos}\text{-}S$ are complete S -posets.*
- (ii) *All S -posets are complete.*

Proof. (i) \implies (ii). Let A_S be an S -poset. Consider the generator $A_S \times S_S$ which is a complete S -poset by assumption. Now, it is easily seen that A_S is complete.

(ii) \implies (i). It is trivial. \square

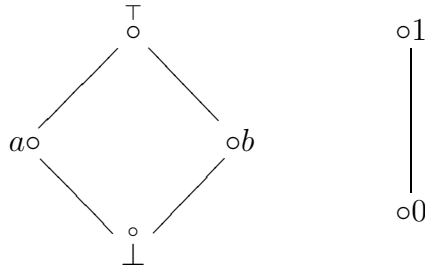
The authors in [9], showed the following proposition which is useful in the sequel.

Proposition 3.7. *Let S be a pomonoid. Suppose $f : A_S \rightarrow B_S$ is an *Emb-injective* object in $\mathbf{Pos}\text{-}S/B_S$ for any S -poset B_S . Then f is a split epimorphism.*

Corollary 3.8. *Suppose $f : A_S \rightarrow B_S$ is an *Emb-injective* object in $\mathbf{Pos}\text{-}S/B_S$. If A is a complete lattice which is a retract of $A^{(S)}$, then A_S and B_S are *Emb-injective* in $\mathbf{Pos}\text{-}S$.*

Proof. By hypothesis we conclude that A_S is an *Emb-injective* S -poset (see [5]). Also, f is a split epimorphism (Proposition 3.7), consequently B_S as a retract of an *Emb-injective* is an *Emb-injective* S -poset. \square

Remark 3.9. There exists a split epimorphism in $\mathbf{Pos}\text{-}S$ which is not *Emb-injective* in $\mathbf{Pos}\text{-}S/B_S$. To present an example, take an arbitrary pomonoid S and X, B are two lattices as shown in the following



Then X is an S -poset with the action defined by $\top s = \top$ and $as = bs = \perp s = a$ for all $s \in S$, also we consider B with the trivial action as an S -poset. Define the S -poset map $f : X_S \rightarrow B_S \in \mathbf{Pos} - S/B$, by $f(a) = f(b) = f(\perp) = 0$ and $f(\top) = 1$. It also is a convex map. We show that it is not a regular injective object in $\mathbf{Pos} - S/B_S$. Since $f^{-1}(0) = \{\perp, a, b\}$ is not a complete lattice, the authors in [7], showed that it is not Emb-injective in $\mathbf{Pos} - S/B_S$.

On the other hands, define the S -poset map $g : B_S \rightarrow X_S \in \mathbf{Pos} - S/B_S$, by $g(1) = \top, g(0) = g(\perp)$. Then we have $fg = \text{id}_B$, so f is split epimorphism. Therefore, the converse of the above theorem is not true generally.

At the rest of this section, we investigate some connections between Emb-injectivity in $\mathbf{Pos} - S/B_S$ and generators and cyclic projectives in $\mathbf{Pos} - S$.

Theorem 3.10. *If $f : A_S \rightarrow B_S$ is an Emb-injective object in $\mathbf{Pos} - S/B_S$ and B_S is a generator S -poset then A_S is a generator. Further, $_{\text{End}(A_S)}A$ is a cyclic projective in $\text{End}(A_S)\text{-Pos}$.*

Proof. Since $f : A_S \rightarrow B_S$ is Emb-injective so by Proposition 3.7, there exists $g : B_S \rightarrow A_S$ in $\mathbf{Pos} - S$ such that $fg = \text{id}_B$. Also, B_S is a generator in $\mathbf{Pos} - S$ and f is an epimorphism so A_S is a generator (see [11]). Now, applying this fact and Theorem 2.2 from [11], we get that $_{\text{End}(A_S)}A$ is a cyclic projective. \square

Theorem 3.11. *Suppose $f : A_S \rightarrow B_S$ is an Emb-injective object in $\mathbf{Pos} - S/B_S$ and A_S is a cyclic projective S -poset. Then B_S is a cyclic projective S -poset. Moreover, $_{\text{End}(B_S)}B$ is a generator in $\text{End}(B_S)\text{-Pos}$.*

Proof. Since $f : A_S \rightarrow B_S$ is Emb-injective so by Proposition 3.7, there exists $g : B_S \rightarrow A_S$ such that $fg = \text{id}_B$. Also, A_S is a cyclic projective in $\mathbf{Pos} - S$ hence by Proposition 2.1, there exist two S -poset maps $S_S \xrightleftharpoons[\gamma]{\pi} A_S$ such that $\pi\gamma = \text{id}_A$. Then we get $f\pi\gamma g = \text{id}_B$, we get B_S is a cyclic projective by Proposition 2.1. Now by Proposition 3.1 from [11], we conclude that $_{\text{End}(B_S)}B$ is a generator. \square

Theorem 3.12. *Suppose $f : A_S \rightarrow B_S$ is an Emb-injective object in $\mathbf{Pos}\text{-}S/B_S$. Then*

- (i) $\mathbf{Pos}_S(B_S, A_S)$ is a generator in $\mathbf{Pos}\text{-}\mathbf{End}(B_S)$.
- (ii) $\mathbf{Pos}_S(A_S, B_S)$ is a generator in $\mathbf{End}(B_S)\text{-}\mathbf{Pos}$.
- (iii) $\mathbf{Pos}_S(B_S, A_S)$ is a cyclic projective in $\mathbf{End}(A_S)\text{-}\mathbf{Pos}$.
- (iv) $\mathbf{Pos}_S(A_S, B_S)$ is a cyclic projective in $\mathbf{Pos}\text{-}\mathbf{End}(A_S)$.

Proof. Since $f : A_S \rightarrow B_S$ is Emb-injective, in view of Proposition 3.7, there exists $g : B_S \rightarrow A_S$ such that $fg = \text{id}_B$. Applying the functors $\mathbf{Pos}_S(B_S, -)$, $\mathbf{Pos}_S(-, B_S)$, $\mathbf{Pos}_S(-, A_S)$ and $\mathbf{Pos}_S(A_S, -)$ we get the assertions (i), (ii), (iii) and (iv) respectively. \square

Proposition 3.13. *In any of the following cases $\mathbf{Pos}_S(A_S \times B_S, B_S)$ is a generator in $\mathbf{End}(B_S)\text{-}\mathbf{Pos}$, for every $B_S \in \mathbf{Pos}\text{-}S$.*

- (i) A_S is an Emb-injective S -poset.
- (ii) $f : A_S \rightarrow B_S$ is an Emb-injective object in $\mathbf{Pos}\text{-}S/B_S$.

Proof. (i) Consider the projection S -poset map $\pi_B : A \times B \rightarrow B_S$. The authors in [7] showed that it is an Emb-injective object in $\mathbf{Pos}\text{-}S/B_S$. Consequently, by Theorem 3.12(ii), we get the result.

(ii) By Proposition 3.7, there exists $g : B_S \rightarrow A_S$ such that $fg = \text{id}_B$. Consider the unique S -poset map $\varphi_B : B_S \rightarrow A \times B$ to product in such away that $\pi_B \varphi_B = \text{id}_B$. Applying the functor $\mathbf{Pos}_S(-, B_S)$ we obtain

$$\mathbf{End}(B_S) = \mathbf{Pos}_S(B, B) \begin{array}{c} \xrightarrow{\pi_B} \\ \xleftarrow{\varphi_B} \end{array} \mathbf{Pos}_S(A \times B, B)$$

So we have $\bar{\varphi}_B \pi_B = \text{id}_{\mathbf{End}(B_S)}$. This means that $\mathbf{End}(B_S)$ is a retract of $\mathbf{Pos}_S(A \times B, B)$ as we needed (see Theorem 2.1 of [11]). \square

Proposition 3.14. *Let $B_S \in \mathbf{Pos}\text{-}S$ and ${}_T A_S$ be an T - S -biposet and $A \times B$ be a cyclic projective S -poset. If $f : A_S \rightarrow B_S$ is an Emb-injective object in $\mathbf{Pos}\text{-}S/B_S$ and $\lambda : T \rightarrow \mathbf{End}(A_S)$ is an isomorphism then ${}_T A$ is a generator in $T\text{-}\mathbf{Pos}$.*

Proof. Consider the projection $\pi_A : A \times B \rightarrow A_S$ and the unique S -poset map $\varphi_A : A_S \rightarrow A \times B$ to product in such away that $\pi_A \varphi_A = \text{id}_A$. On the other hands, $A \times B$ is cyclic projective S -poset so there exist

two S -poset maps $A \times B \xrightleftharpoons[\pi]{\gamma} S_S$ such that $\pi\gamma = 1_{A \times B}$. Applying the functor $\mathbf{Pos}_S(-, A_S)$ and the composition $\pi_A \pi \gamma \varphi_A = 1_A$, we obtain

$$T \cong Pos_S(A, A) \xrightleftharpoons[\varphi_A]{\pi_A} Pos_S(A \times B, A) \xrightleftharpoons[\bar{\gamma}]{\bar{\pi}} Pos_S(S, A) \cong_T A$$

Hence, T is a retract of $_T A$, so $_T A$ is a generator in $\mathbf{Pos}\text{-}S$. \square

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